

*Positive modal logic and its uses in the study  
of provability in arithmetic*

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## *Strictly positive modal formulas*

The language of modal logic extends that of propositional calculus by a family of unary connectives  $\{\diamond_i : i \in I\}$ .

*Strictly positive* modal formulas are defined by the grammar:

$$A ::= p \mid \top \mid (A \wedge B) \mid \diamond_i A, \quad i \in I.$$

We are interested in the implications  $A \rightarrow B$  where  $A$  and  $B$  are strictly positive.

## *Strictly positive logics*

- *Strictly positive fragment* of a modal logic  $L$  is the set of all implications  $A \rightarrow B$  such that  $A$  and  $B$  are strictly positive and  $L \vdash A \rightarrow B$ .
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## *Sources of interest*

The interest towards strictly positive (fragments of) modal logics independently emerged around 2010 in two different disciplines:

- The work on proof-theoretic applications of *provability logic* by myself, Dashkov, et al.;
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## *Advantages*

- Strictly positive fragment of a modal logic is (in the considered cases) *much simpler* than the original logic.
- Typically, strictly positive fragments of standard modal logics are polytime decidable.

On the other hand, the language is *sufficiently expressive*:

- Many applications of provability logic to the analysis of arithmetical theories can be stated in terms of strictly positive logics.
- Strictly positive logics allow for alternative arithmetical interpretations that are quite natural from a proof-theoretic point of view.



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## Part I: Basic strictly positive logic

We derive sequents of the form  $A \vdash B$  with  $A, B$  positive.<sup>1</sup>

$\mathbf{K}^+$ : the positive fragment of  $\mathbf{K}$

- ①  $A \vdash A$ ;  $A \vdash \top$ ; if  $A \vdash B$  and  $B \vdash C$  then  $A \vdash C$ ;
- ②  $A \wedge B \vdash A, B$ ; if  $A \vdash B$  and  $A \vdash C$  then  $A \vdash B \wedge C$ ;
- ③ if  $A \vdash B$  then  $\diamond A \vdash \diamond B$ .

**Fact.**  $\mathbf{K}^+$  is closed under substitution and positive replacement:

- if  $A(p) \vdash B(p)$  then  $A(C) \vdash B(C)$ ;
- if  $A \vdash B$  then  $C(A) \vdash C(B)$ .

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## Formulas as trees

To every positive formula  $A$  there corresponds its minimal Kripke model  $T[A]$  which I call its *canonical tree*. It is morally the same as its parse tree.

For any rooted Kripke model  $M$ , we write  $M \Vdash B$  to mean  $B$  is true at the root of  $M$ .

*Fact 1.*  $A \vdash_{K^+} B$  iff  $T[A] \Vdash B$ .

*Fact 2.* For any rooted Kripke model  $M$ ,  $M \Vdash B$  iff there is a root-preserving homomorphism of  $T[B]$  into  $M$ .

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## Normal positive logics

A *normal positive logic* is a set of sequents closed under the rules of  $K^+$  and the substitution rule.

Other standard logics (with Daskov, Svyatlovsky):

$$(4) \quad \Diamond\Diamond A \vdash \Diamond A;$$

$$(T) \quad A \vdash \Diamond A;$$

$$(5) \quad \Diamond A \wedge \Diamond B \vdash \Diamond(A \wedge \Diamond B).$$

*Fact (Dashkov).* Positive fragment of GL coincides with  $K4^+$ .

**Warning!** Positive fragments of normal modal logics are normal, but not every normal positive logic is a fragment of some normal modal logic.



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## String rewriting (semi-Thue) systems

Fix an alphabet  $\Sigma$ .

Interpret a word  $A = a_1 a_2 \dots a_n \in \Sigma^*$  as the positive formula  $A^* = \diamond_{a_1} \diamond_{a_2} \dots \diamond_{a_n} p$  ( $p$  a distinguished variable)

Rewrite rule  $A \rightarrow B$  corresponds to  $A^* \vdash B^*$ .

*Th.* To any string rewriting system  $R$  in  $\Sigma$  we can naturally associate a positive normal logic  $L_R$  such that  $A \rightarrow_R^* B$  iff  $A^* \vdash_{L_R} B^*$ .

*Cor.* There are undecidable positive normal logics.

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## Basic deep inference calculus

**Rules:**

$$\frac{A}{A \wedge A} \quad \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B} \quad \frac{A}{\top}$$

*Context:* A formula  $C(p)$  with a unique occurrence of a distinguished variable  $p$ .

*Rule application:* Replace  $C(A)$  by  $C(B)$ , where  $C(p)$  is any context and  $\frac{A}{B}$  is a rule instance.

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## *Part II: Reflection principles*

Emerged in the 1930s in the work of Rosser, Kleene and Turing. Later (in the 1960s) it was taken up by Kreisel, Levy, Feferman and others.

Cf. G. Kreisel and A. Levy (1968): *Reflection principles and their use for establishing the complexity of axiomatic systems.*

## Reflection principles

Notation:

$\Box_S(\varphi)$       ' $\varphi$  is provable in  $S$ '

$Tr_n(\sigma)$       ' $\sigma$  is the Gödel number of a true  $\Sigma_n$ -sentence'

Restricted reflection principles:

$R_0(S)$        $\text{Con}(S)$

$R_n(S)$        $\forall \sigma \in \Sigma_n (\Box_S \sigma \rightarrow Tr_n(\sigma))$ , for  $n \geq 1$ .

$R_n(S)$  can be seen as a relativization of the consistency assertion:

$R_n(S) \iff \text{Con}(S + \text{all true } \Pi_n\text{-sentences})$



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## *Unrestricted reflection*

- Uniform reflection:  $R_\omega(S) := \{R_n(S) : n \in \omega\}$ .  
 $\{\forall x (\Box_S \varphi(\dot{x}) \rightarrow \varphi(x)) : \varphi(x) \text{ any arithmetical formula}\}$
- Local reflection:  
 $\{\Box_S \varphi \rightarrow \varphi : \varphi \text{ any arithmetical sentence}\}$

Neither of the two schemata is finitely axiomatizable.  $R_\omega(S)$  is not contained in any consistent extension of  $S$  of bounded arithmetical complexity.

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## Reflection algebra of $S$

Let  $\mathcal{L}_S$  be the Lindenbaum–Tarski boolean algebra of  $S$  sentences.

- Each  $R_n$  correctly defines an operator on the equivalence classes of  $\mathcal{L}_S$ :  $\langle n \rangle : [\varphi] \mapsto [R_n(S + \varphi)]$ .
- The algebra  $(\mathcal{L}_S, \langle 0 \rangle, \langle 1 \rangle, \dots)$  is the *reflection algebra of  $S$* .

Identities of this algebra are axiomatized by the polymodal provability logic  $GLP$  due to Japaridze.

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## *GLP: a Hilbert-style calculus*

Basic symbols are now  $[n]$ , for each  $n \in \omega$ , and  $\langle n \rangle$  is treated as an abbreviation:  $\langle n \rangle \varphi = \neg[n]\neg\varphi$ .

- 1 Tautologies;
- 2  $[n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi)$ ;
- 3  $[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$ ;
- 4  $[n]\varphi \rightarrow [n+1]\varphi$ ;
- 5  $\langle n \rangle \varphi \rightarrow [n+1]\langle n \rangle \varphi$

Rules: modus ponens,  $\varphi \vdash [n]\varphi$ .

**Th.** (Japaridze, 1986)  $\text{GLP} \vdash \varphi(\vec{x})$  iff  $\mathcal{L}_S \models \forall \vec{x} (\varphi(\vec{x}) = 1)$ , provided  $\mathbb{N} \models S$ .

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## Reflection calculus RC

Language:  $\alpha ::= \top \mid p \mid (\alpha_1 \wedge \alpha_2) \mid n\alpha \quad n \in \omega$

Example:  $\alpha = 3(2p \wedge 32\top)$ , or shortly:  $3(2p \wedge 32)$ .

Sequents:  $\alpha \vdash \beta$ .

RC rules:

- 1  $\alpha \vdash \alpha$ ;  $\alpha \vdash \top$ ; if  $\alpha \vdash \beta$  and  $\beta \vdash \gamma$  then  $\alpha \vdash \gamma$ ;
- 2  $\alpha \wedge \beta \vdash \alpha, \beta$ ; if  $\alpha \vdash \beta$  and  $\alpha \vdash \gamma$  then  $\alpha \vdash \beta \wedge \gamma$ ;
- 3  $n\alpha \vdash n\alpha$ ; if  $\alpha \vdash \beta$  then  $n\alpha \vdash n\beta$ ;
- 4  $n\alpha \vdash m\alpha$  for  $n > m$ ;
- 5  $n\alpha \wedge m\beta \vdash n(\alpha \wedge m\beta)$  for  $n > m$ .

Ex.  $3 \wedge 23 \vdash 3(\top \wedge 23) \vdash 323$ .



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## *Interpretation of RC in GLP*

- RC can be seen as a strictly positive fragment of GLP.
- Interpretation:  $3(2p \wedge 32T) \mapsto \langle 3 \rangle (\langle 2 \rangle p \wedge \langle 3 \rangle \langle 2 \rangle T)$

### **Theorems** (E. Dashkov).

- 1 GLP is a conservative extension of RC;
- 2 RC is polytime decidable;
- 3 RC enjoys the finite model property.

## $RC^0$ as an ordinal notation system

Let  $RC^0$  denote the variable-free fragment of  $RC$ .

Let  $W$  denote the set of all  $RC^0$ -formulas. For  $\alpha, \beta \in W$  define:

- $\alpha \sim \beta$  if  $\alpha \vdash \beta$  and  $\beta \vdash \alpha$  in  $RC^0$ ;
- $\alpha <_n \beta$  if  $\beta \vdash n\alpha$ .

### Theorem.

- 1 Every  $\alpha \in W$  is equivalent to a *word* (formula without  $\wedge$ );
- 2  $(W/\sim, <_0)$  is isomorphic to  $(\varepsilon_0, <)$ .

**Rem.**  $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$  is the characteristic ordinal of Peano arithmetic.

## The system $RC_\omega$

The language of  $RC$  is extended by a new modality symbol  $\omega$ . The rules of  $RC_\omega$  are:

- The rules of  $RC$  stated for all  $n \leq \omega$ ;
- $\omega\alpha \vdash \alpha$ .

The intended interpretation of  $\omega$  is  $R_\omega$ . Hence, the strictly positive modal formulas should now be understood as (possibly infinite) *sets* of arithmetical sentences.

## *Axiom $\omega\alpha \vdash \alpha$*

- Modal logically, this schema means  $\Diamond A \rightarrow A$ , i.e.,  $A \rightarrow \Box A$ .
- Notice that in the context of classical modal logic the principle  $A \rightarrow \Box A$  only has trivial (discrete) Kripke frames.
- Not so in the strictly positive logic! The same principle also plays a certain role in intuitionistic modal logic.

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## Arithmetical interpretation of $RC\omega$

Let  $S$  be a gödelian theory.

- An *arithmetical substitution*  $\sigma$  assigns to each variable a gödelian theory extending  $S$  (i.e. a primitive recursive set of sentences together with a p.r. formula defining this set).
- *Interpretation*  $\alpha^\sigma$  of a strictly positive formula  $\alpha$  in  $S$ :
  - $\top^\sigma = \emptyset$ ;  $(\alpha \wedge \beta)^\sigma = (\alpha^\sigma \cup \beta^\sigma)$ ;
  - $(n\alpha)^\sigma = \{R_n(S + \alpha^\sigma)\}$ ;
  - $(\omega\alpha)^\sigma = R_\omega(S + \alpha^\sigma)$ .



# Arithmetical completeness

Suppose  $\mathbb{N} \models S$ .

**Theorem 1.**  $\alpha \vdash \beta$  in  $RC\omega$  iff  $\forall \sigma S + \alpha^\sigma \vdash \beta^\sigma$ .

**Theorem 2.**  $RC\omega$  enjoys the finite model property and is polytime decidable.

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## Example

- A simple Kripke model shows that

$$\omega p \wedge \omega q \not\vdash_{RC\omega} \omega(p \wedge q).$$

By Theorem 1, there are arithmetical theories  $P, Q$  such that

$$S + R_\omega(P) + R_\omega(Q) \not\vdash R_\omega(P + Q).$$

- It is easy to see that any such pair must have unrestricted arithmetical complexity over  $S$ .

## $RC^0_\omega$ as an ordinal notation system

Let  $RC^0_\omega$  denote the variable-free fragment of  $RC_\omega$ .

Let  $W_\omega$  denote the set of all  $RC^0_\omega$ -formulas.

**Theorem 3.** In  $RC^0_\omega$ ,

- 1 Every  $\alpha \in W_\omega$  is equivalent to a *word* (formula without  $\wedge$ );
- 2  $(W_\omega/\sim, <_0)$  is isomorphic to  $(\varepsilon_\omega, <)$ .

Here  $\varepsilon_\omega$  is the  $\omega$ -th ordinal  $\alpha$  such that  $\omega^\alpha = \alpha$ .

## *Open questions*

- Study systematically strictly positive normal logics (*s.p.l.*). State general sufficient conditions guaranteeing nice semantic properties of such logics.
- Characterize strictly positive fragments of standard polymodal logics by positive calculi. (Some preliminary work done by Dashkov, Kaniskin and more to come.)
- Study the complexity of s.p.l.. (Need not always be polytime decidable or even decidable.)
- Study the analogy between s.p.l. and semi-Thue systems.
- Positive Craig interpolation?

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## *Papers*

- E. Dashkov. *On the positive fragment of the polymodal provability logic GLP*. Mathematical Notes, 2012, Vol. 91, No. 3, pp. 318–333.
- L. Beklemishev. *Positive provability logic for uniform reflection principles*. Annals of Pure and Applied Logic, published online August 2013. DOI: [10.1016/j.apal.2013.07.006](https://doi.org/10.1016/j.apal.2013.07.006)