Sequential Theories

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Universiteit Utrecht

Overview

Sequential Theories

Satisfaction and Restricted Consistency

Generalization of the Orey-Hájek Characterization

It's hard to keep Finiteness out

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What is this talk about?

The aim of this talk is twofold. First we introduce an important class of theories, *the sequential theories*, that supports significant theorems. This class of theories generalizes the classes of arithmetical theories and of set theories. It ranges from very weak to very strong theories. We will give the basic definitions and describe a fundamental result: *the generalization of the Orey-Hájek Theorem*. Secondly, we present a limitative result.

In model theory, people studied classes of good theories like the o-minimal ones. Such theories have decent, tame behavior. Sequential theories are far from tame. For example, they are all essentially undecidable. Yet, as we will see, also such theories may share salient exciting global features.

One of the themes of this conference is *limits*. We know that predicate logic cannot define finiteness. In the second half of this talk we will show that under certain conditions it is hard to avoid the presence of a model in which the natural numbers are definable.

What is a Sequential Theory? 1

Sequential theories are theories with sequence coding. We are able to talk of sequences *of variable length*. The sequences may be formed of *any* objects of the theory.

The notion of sequentiality is inextricably connected with the possibility to build, internally, *partial satisfaction predicates* and with the possibility to extend models, externally, with e.g. full satisfaction predicates. The machinery present in a sequential theory is more or less precisely what one needs to formulate what such a predicate does.

A consequence of the presence of partial satisfaction predicates is that we can prove restricted consistency statements for our theories.

What is a Sequential Theory? 2

Sequential theories have very simple definition due to Pavel Pudlák. The theory AS is given by the following axioms.

AS1 $\vdash \exists y \forall x \; x \notin y$,

AS2 $\vdash \forall x \, \forall y \, \exists z \forall u \, (u \in z \leftrightarrow (u \in y \vee u = x))$.

Note that we do not have *extensionality*.

A theory is sequential when the theory AS is directly interpretable in it. This means that there is a formula $E(x, y)$ in the theory for which the axioms of AS can be verified.

We can with minor adaptations work with the broader notion of *polysequentiality* where we allow our 'sets' and 'elements' to be *n*-ary sequences of objects.

What is a Sequential Theory? 3

By a gigantic bootstrap we can show that:

- In a sequential theory we have an interpretation of *number theory*. Specifically we can interpret Buss' weak arithmetic S^1_2. In this theory we can develop syntax in the usual way. We can verify the Second Incompleteness Theorem. Internalization of Rosser style arguments becomes problematic.
- In a sequential theory we can find a better virtual class of sets that is closed under empty sets, singletons, unions, subtractions, cartesian products. Still the new sets are sets for all objects.
- We can develop number theory from the sets following one of the different strategies known from ordinary set theory enriched by an array of tricks to make arguments work where we do not have the strength of ordinary set theory available.
- We can use the better sets to form a theory of sequences with closure under empty sequences, unit sequences, concatenation, local reset. These sequences have projections in our chosen 'numbers'.

What is a Sequential Theory? 4

We illustrate the big bootstrap by showing how to get closure under union —the very first step in the process. The basic idea for the big bootstrap comes from Solovay.

We work in AS. Consider the virtual class U of all sets x such that for all y a union $x \cup y$ exists. It is easy to see that empty sets and singletons are in U.

Suppose x_0 and x_1 are in U. Consider any y. Since x_1 is in U, there is a set $x_1 \cup y$. Since x_0 is in U, there are sets $x_0 \cup x_1$ and $x_0 \cup (x_1 \cup y)$. Clearly, the set $x_0 \cup (x_1 \cup y)$ also plays the role of a set $(x_0 \cup x_1) \cup y$, Hence, $(x_0 \cup x_1)$ exists and is in U.

Warning: any two sets in U may have many different ‘unions’.

What is a Sequential Theory? 5

- Adjunctive Set Theory AS.
- PA^-, the theory of discretely ordered commutative semirings with a least element.
- Buss' theory S^1_2.
- Wilkie and Paris' theory $I\Delta_0 + \Omega_1$.
- Elementary Arithmetic EA (aka Elementary Function Arithmetic EFA, or $I\Delta_0 + \exp$).
- PRA.
- $I\Sigma^0_1$.
- Peano Arithmetic PA.
- ACA_0.
- ZF.
- GB.

Non-examples: Presburger Arithmetic, theory of pairing, Vaught Set Theory, Robinson's Arithmetic Q, Real Closed Fields.

What is a Sequential Theory? 6

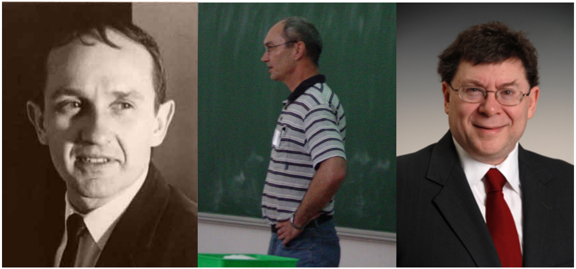

Figure: Montague (1961), Pudlák (1983), Friedman (1985)

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Formula Classes

We give a hierarchy for *depth of quantifier alternations*. Let a signature Θ be given. We define our classes $\Sigma^*_n(\Theta)$ and $\Pi^*_n(\Theta)$ as follows (suppressing the Θ):

- $\Sigma^*_0 := \Pi^*_0 := \emptyset$.
- $\Sigma^*_{n+1} ::= \mathrm{AT} \mid \neg\, \Pi^*_{n+1} \mid (\Sigma^*_{n+1} \wedge \Sigma^*_{n+1}) \mid (\Sigma^*_{n+1} \vee \Sigma^*_{n+1}) \mid (\Pi^*_{n+1} \rightarrow \Sigma^*_{n+1}) \mid \exists v\, \Sigma^*_{n+1} \mid \forall v\, \Pi^*_n$.
- $\Pi^*_{n+1} ::= \mathrm{AT} \mid \neg\, \Sigma^*_{n+1} \mid (\Pi^*_{n+1} \wedge \Pi^*_{n+1}) \mid (\Pi^*_{n+1} \vee \Pi^*_{n+1}) \mid (\Sigma^*_{n+1} \rightarrow \Pi^*_{n+1}) \mid \forall v\, \Pi^*_{n+1} \mid \exists v\, \Sigma^*_n$.

$\rho(A)$ is the smallest k such that $A \in \Sigma^*_k$.

Our definition is an adaptation of a definition of Sam Buss.

Satisfaction

We can build satisfaction predicates Sat_n for the $\Sigma_n^*(\Theta)$ in a given sequential theory U. These predicates verifiably satisfy the commutation conditions in a definable cut I of the given numbers N. E.g. we have:

$$U \vdash \forall B, C \in (\Sigma_n^* \cap I)\, \forall\alpha\, (\mathsf{Sat}_n(\alpha, B \wedge C) \leftrightarrow (\mathsf{Sat}_n(\alpha, B) \wedge \mathsf{Sat}_n(\alpha, C))).$$

We write $\mathsf{True}_n(A)$ for $\forall\alpha\, \mathsf{Sat}_n(\alpha, A)$ in case A is a sentence.

Relativization to a definable cut is necessary to compensate for the lack of induction.

Reflection

We write $\vdash_{U,n}$ or provability that only involves Σ_n^*-formulas where the (codes of the) U-axioms used are below n. We write $\mathrm{prov}_{U,n}$ for the formalized version of $\vdash_{U,n}$.

We can define a cut J of N such that:

$$U \vdash \forall A \in \Sigma_n^* \cap J\ (\mathrm{prov}_{U,n}^J A \rightarrow \mathrm{True}_n(A))$$

It follows that, for all n, there is a cut J such that $U \vdash \mathrm{con}_n^J(U)$. Thus, sequential theories are *locally reflexive*.

By a variant of the Second Incompleteness Theorem, we have: for all cuts J, there is an n, such that $U \nvdash \mathrm{con}_n^J(U)$.

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What is Interpretability?

One theory U is *interpretable* in another theory V if there is a translation τ such that, for all U-sentences A, if $U \vdash A$ then $V \vdash A^\tau$.

What is a *translation*? As a first approximation, we can say: anything that commutes with the predicate logical connectives. An n-ary U-predicate P will be translated to a V-formula $A(x_0, \ldots, x_{n-1})$.

We allow *domain-relativization*: $\forall x\, Bx$ is translated to $\forall x\, (\delta(x) \to B^\tau x)$.

There are more refinements that we blissfully ignore here.

- $V \rhd U$ for V interprets U.
- $V \rhd_{\mathrm{loc}} U$ for V locally interprets U, i.e., for all finitely axiomatized subtheories U_0 of U we have $V \rhd U_0$.

The mho-functor

Let U be recursively enumerable and sequential.

- $\mho(U) := \mathsf{S}^1_2 + \{\mathsf{con}_n(U) \mid n \in \omega\}$.

The fact that U is locally reflexive can now be formulated as $U \rhd_{\mathsf{loc}} \mho(U)$.

Combining the Gödel-Hilbert-Bernays-Wang-Henkin-Feferman theorem with an observation due to Feferman, we find: $\mho(V) \rhd V$.

This gives us $U \rhd_{\mathsf{loc}} V$ iff $\mho(U) \vdash \mho(V)$. It follows that:

$$(\dagger) \quad V \lhd_{\mathsf{loc}} U \quad \Leftrightarrow \quad V \lhd \mho(U).$$

This tells us that $\mho$ is the right adjoint of the projection functor of the degrees of global interpretability in the degrees of local interpretability. Note that $\mho$ is uniquely determined modulo mutual interpretability by equation ($\dagger$).

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The Existence Property

Let k be any number. Consider a sequential theory U such that all axioms of U are in Σ_k^*. There are no restrictions on the axiom set of U. Then there is a cut J such that for any sentence $\exists x \in N\, B_0(x)$ with complexity below k, we have:

Suppose $U \vdash \exists x \in J\, B_0(x)$. Then, for some k, we have $U \vdash \exists x \leq^N \underline{k}\, B_0(x)$, or, equivalently, $U \vdash \bigvee_{q \leq k} B_0(\underline{q})$.

Even if the statement of the result has a strongly proof-theoretic flavor, we used a Rosser argument to prove it.

A Lindström-Style Result

A theory U is *restricted* if for some n all axioms have complexity below n.

Suppose U is a restricted consistent, recursively enumerable, sequential theory. Let m be any number. Then there is a finitely axiomatized theory A in the same language such that A extends U and is m-conservative over U.

For example, for any recursively enumerable extension U of PA there is a finite extension A of ACA_0 such that A has the same arithmetical consequences as U. (Robert van Wesep, 2012)

MacAloon

Kenneth MacAloon (1978) proves that for any n, there is a Π_{n+2}-formula $\mathcal{I}$ such that any countable model of PA has a Π_n-elementary end-extension in which $\mathcal{I}$ defines a cut which contains precisely the natural numbers (modulo isomorphism).

For any sequential model $\mathcal{M}$ we define $\mathcal{J}_{\mathcal{M}}$ as the intersection of all definable cuts. One can show that $\mathcal{J}_{\mathcal{M}}$ is modulo isomorphism independent of the class of numbers given.

For any n, any sequential model $\mathcal{M}$ has a Σ_n^*-elementary extension $\mathcal{K}$ such that $\mathcal{J}_{\mathcal{K}}$ is the standard numbers.

Our result is at the same time far more general and substantially weaker than MacAloon's. It remains open whether we can always find a Σ_n^*-elementary extension in which the standard natural numbers are actually definable.

Thank You

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